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ON THE LOG-CONVEXITY OF THE DIFFERENCE SEQUENCE OF A LOG-CONVEX SEQUENCE

FENG-ZHEN ZHAO

ABSTRACT. Let $\{z_n\}_{n\geq 0}$ be a log-convex sequence, where $z_{n+1}-z_n>0$ for $n\geq 0$. In this paper, we give several sufficient conditions for the log-convexity of the sequence $\{z_{n+1}-z_n\}_{n\geq 0}$. As applications, we discuss log-convexity for a series of sequences involving derangement numbers, Motzkin numbers, Fine numbers, numbers of counting directed animals, and so on.

1. INTRODUCTION

For convenience, we first recall some definitions for the log-behavior of a sequence in combinatorics. For a positive sequence $\{z_n\}_{n\geq 0}$, it is said to be *log-convex* (*log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ ($z_n^2 \geq z_{n-1}z_{n+1}$) for all $n\geq 1$. Log-behavior of sequences plays an important role in many subjects. In particular, log-behavior is one of the fertile sources of combinatorial inequalities. Hence the log-behavior of sequences deserves to be studied. In this paper, we consider the log-convexity of sequences. Many famous combinatorial sequences, including Catalan numbers, derangement numbers, Motzkin numbers, Fine numbers, large Schröder numbers, and central Delannoy numbers, are log-convex respectively. There are some operators preserving log-convexity. If both $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are log-convex sequences, then so is the sequence $\{u_n+v_n\}_{n\geq 0}$ (For more operators preserving log-convexity, see Liu and Wang [6]). The sequence $\{u_n-v_n\}_{n\geq 0}$ is not log-convex in general, where $u_n-v_n>0$ for $n\geq 0$. For example, for the Fibonacci sequence $\{F_n\}_{n\geq 0}$, where $F_0=0$ and $F_1=1$, it is well known that $\{F_{2n+1}\}_{n\geq 0}$ is log-convex, but $\{F_{2n+3}-F_{2n+1}\}_{n\geq 0}=\{F_{2n+2}\}_{n\geq 0}$ is log-concave. The aim of this paper is to study the log-convexity of the sequence $\{z_{n+1}-z_n\}_{n\geq 0}$, where $\{z_n\}_{n\geq 0}$ is log-convex and $z_{n+1}-z_n>0$ for $n\geq 0$. In the next section, we give several sufficient conditions for log-convexity of the sequence $\{z_{n+1}-z_n\}_{n\geq 0}$, where $\{z_n\}_{n\geq 0}$ is log-convex and satisfies three-term linear recurrences. As applications, we discuss log-convexity for a series of sequences involving derangement numbers, Motzkin numbers, Fine numbers, numbers of counting directed animals, and so on.

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2. SEVERAL SUFFICIENT CONDITIONS FOR THE LOG-CONVEXITY OF THE DIFFERENCE SEQUENCE OF A LOG-CONVEX SEQUENCE

In this section, we give the main results of this paper.

Theorem 2.1. *Suppose that* $\{z_n\}_{n\geq 0}$ *is a log-convex sequence defined by*

$$z_{n+1}=R(n)z_n+S(n)z_{n-1}, \quad n\geq 1, \tag{2.1}$$

where $R(n)>1$ *and* $S(n)\geq 0$ *for each* $n\geq 1$*. For* $n\geq 0$*, let* $x_n=\frac{z_{n+1}}{z_n}$ *and*

$$\begin{aligned} f(t) &= R(n+2)-R(n+1)+\frac{R(n+2)+S(n+2)-1}{[R(n+1)-1]t+S(n+1)}t \\ &\quad -\frac{R(n+1)+S(n+1)-1}{t-1}, \end{aligned}$$

where $t>1$*. Assume that*

$$\phi_n\leq x_n, \quad n\geq N_0,$$

where N_0 *is an integer with* $N_0\geq 0$ *and* $\phi_n>1$ *for* $n\geq N_0$*. If there exists an integer* $N\geq N_0$ *such that* $f(\phi_n)\geq 0$ *for* $n\geq N$*, the sequence* $\{z_{n+1}-z_n\}_{n\geq N}$ *is log-convex.*

Proof. For $n\geq N_0$, let $y_n=\frac{z_{n+2}-z_{n+1}}{z_{n+1}-z_n}$. In order to show that $\{z_{n+1}-z_n\}_{n\geq N}$ is log-convex, we need to prove that $\{y_n\}_{n\geq N}$ is increasing. Since $\{z_n\}_{n\geq 0}$ is log-convex, $\{x_n\}_{n\geq 0}$ is increasing. On the other hand, $\phi_n>1$ ($n\geq N_0$). Then $z_{n+1}-z_n>0$ and $x_n-1>0$ for $n\geq N_0$. By applying (2.1), we derive

$$x_n = R(n)+\frac{S(n)}{x_{n-1}}, \quad n\geq 1, \tag{2.2}$$

$$\begin{aligned} z_{n+1}-z_n &= [R(n)-1]z_n+S(n)z_{n-1} \quad (n\geq 1) \\ &= [R(n)-1](z_n-z_{n-1})+[R(n)-1+S(n)]z_{n-1}. \end{aligned} \tag{2.3}$$

By means of (2.3), we get

$$\begin{aligned} y_{n+1}-y_n &= R(n+2)-R(n+1)+\frac{R(n+2)-1+S(n+2)}{x_{n+1}-1} \\ &\quad -\frac{R(n+1)-1+S(n+1)}{x_n-1}, \end{aligned}$$

where $n\geq N_0$. By using (2.2), we have

$$\begin{aligned} y_{n+1}-y_n &= R(n+2)-R(n+1)+\frac{R(n+2)-1+S(n+2)}{[R(n+1)-1]x_n+S(n+1)}x_n \\ &\quad -\frac{R(n+1)-1+S(n+1)}{x_n-1} \\ &= f(x_n) \quad (n\geq N). \end{aligned}$$

Since $f'(t)>0$, $f(t)$ is increasing on $(1,+\infty)$. Then $f(x_n)\geq f(\phi_n)\geq 0$ ($n\geq N$). Hence the sequence $\{y_n\}_{n\geq N}$ is increasing. $\square$

Theorem 2.2. *Suppose that $\{z_n\}$ is a log-convex sequence defined by*

$$z_{n+1} = R(n)z_n - S(n)z_{n-1}, \quad n \geq 1, \tag{2.4}$$

where $R(n) > 1$ and $S(n) \geq 0$ for $n \geq 1$. For $n \geq 0$, let $x_n = \frac{z_{n+1}}{z_n}$. Suppose that

$$\phi_n \leq x_n \leq \varphi_n, \quad n \geq N_1,$$

where N_1 is an integer with $N_1 \geq 0$ and $\phi_n > 1$ for $n \geq N_1$.

(i) For $n \geq N_1$, let

$$\begin{aligned}\Delta(n) &= R(n+2) - R(n+1) + \frac{R(n+2) - 1 - S(n+2)}{\varphi_{n+1} - 1} \\ &\quad - \frac{R(n+1) - 1 - S(n+1)}{\phi_n - 1}.\end{aligned}$$

If $R(n) - 1 - S(n) \geq 0$ for $n \geq 1$ and there exists an integer $N_2 \geq N_1$ such that $\Delta(n) \geq 0$ for $n \geq N_2$, the sequence $\{z_{n+1} - z_n\}_{n \geq N_2}$ is log-convex.

(ii) For $n \geq N_1$, let

$$\begin{aligned}\Omega(n) &= R(n+2) - R(n+1) + \frac{S(n+1) + 1 - R(n+1)}{\varphi_n - 1} \\ &\quad - \frac{S(n+2) + 1 - R(n+2)}{\phi_{n+1} - 1}.\end{aligned}$$

If $R(n) \leq S(n)$ for $n \geq 1$ and there exists an integer $N_3 \geq N_2$ such that $\Omega(n) \geq 0$ for $n \geq N_3$, the sequence $\{z_{n+1} - z_n\}_{n \geq N_3}$ is log-convex.

Proof. We only give the proof of (i). The proof of (ii) follows the same pattern and is omitted here. Since $\{z_n\}_{n \geq 0}$ is log-convex and $\phi_n > 1$ $(n \geq N_1)$, we have $z_{n+1} - z_n > 0$ and $x_n - 1 > 0$ for each $n \geq N_1$. It follows from (2.4) that

$$z_{n+1} - z_n = [R(n) - 1](z_n - z_{n-1}) + [R(n) - 1 - S(n)]z_{n-1}. \tag{2.5}$$

For $n \geq N_1$, let $y_n = \frac{z_{n+2} - z_{n+1}}{z_{n+1} - z_n}$. It follows from (2.5) that

$$y_n = R(n+1) - 1 + \frac{R(n+1) - 1 - S(n+1)}{x_n - 1}, \quad (n \geq N_1).$$

Thus we obtain

$$\begin{aligned}y_{n+1} - y_n &= R(n+2) - R(n+1) + \frac{R(n+2) - 1 - S(n+2)}{x_{n+1} - 1} \\ &\quad - \frac{R(n+1) - 1 - S(n+1)}{x_n - 1}, \quad (n \geq N_1).\end{aligned}$$

Since $\phi_n \le x_n \le \varphi_n$ $(n \ge N_1)$, we have

$$\begin{aligned} y_{n+1}-y_n &\ge R(n+2)-R(n+1)+\frac{R(n+2)-1-S(n+2)}{\varphi_{n+1}-1} \\ &\quad -\frac{R(n+1)-1-S(n+1)}{\phi_n-1} \\ &= \Delta(n) \quad (n \ge N_2). \end{aligned}$$

Noting that $\Delta(n) \ge 0$ $(n \ge N_2)$, we obtain $y_{n+1}-y_n \ge 0$ $(n \ge N_2)$. Hence the sequence $\{z_{n+1}-z_n\}_{n\ge N_2}$ is log-convex. □

Now we give the applications of Theorems 2.1–2.2.

Example 2.1. *The sequence of derangement numbers* $\{d_n\}_{n\ge 0}$ *satisfies the following recurrence*

$$d_{n+1} = nd_n + nd_{n-1}, \quad n \ge 1,$$

where $d_0 = 1$ *and* $d_1 = 0$*; see Table 2.1 for more values of* $\{d_n\}_{n\ge 0}$*. The value of* d_n *is the number of permutations of n elements with no fixed points. The sequence* $\{d_n\}_{n\ge 0}$ *is Sloane's A000166 [8]. In particular, Liu and Wang [6] proved that* $\{d_n\}_{n\ge 2}$ *is log-convex.*

n	0	1	2	3	4	5	6	7	8	9
d_n	1	0	1	2	9	44	265	1854	14833	133496

Table 2.1. Some initial values of $\{d_n\}_{n\ge 0}$

Theorem 2.3. *For the sequence of derangement numbers* $\{d_n\}_{n\ge 0}$, $\{d_{n+1}-d_n\}_{n\ge 3}$ *is log-convex.*

Proof. It is evident that

$$\begin{aligned} &R(n) = n, \quad S(n) = n, \\ &f(t) = 1+\frac{(2n+3)t}{nt+n+1}-\frac{2n+1}{t-1}, \quad t>1. \end{aligned}$$

For $n \ge 2$, let $x_n = \frac{d_{n+1}}{d_n}$. Liu and Zhao [7] proved that

$$\eta_n \le x_n, \quad n \ge 3,$$

where $\eta_n = n+\frac{1}{2}$. It is clear that $\eta_n > 1$ for $n \ge 3$. By computation, we derive

$$\begin{aligned} f(\eta_n) &= \frac{(2n+3)(2n^2-3n-3)}{(2n-1)(2n^2+3n+2)} \\ &> 0 \quad (n \ge 3). \end{aligned}$$

It follows from Theorem 2.1 that $\{d_{n+1}-d_n\}_{n\ge 3}$ is log-convex. □

Example 2.2. *A relation* $\Re$ *is called a bipermutation of* $[n] = \{1,2,\cdots,n\}$ *if all vertical sections and all horizontal sections have* 2 *elements. Let* $\{P_n\}_{n\ge 0}$ *denote the sequence for the number of bipermutations. The sequence* $\{P_n\}_{n\ge 0}$ *is Sloane's*

A001499 [8] and satisfies the following recurrence relation

$$P_{n+1} = \binom{n+1}{2}(2P_n + nP_{n-1}), \quad n \geq 1, \tag{2.6}$$

where $P_0 = 1$ and $P_1 = 0$; see Table 2.2 for more values of $\{P_n\}_{n\geq 0}$. There are some properties of $\{P_n\}_{n\geq 0}$ in Comtet [1, p. 235–236]. In particular, Zhao [10] proved that $\{P_n\}_{n\geq 2}$ is log-convex.

n	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
P_n	*1*	*0*	*1*	*6*	*90*	*2040*	*67950*	*3110940*	*187530840*

Table 2.2. *Some initial values of* $\{P_n\}_{n\geq 0}$

Theorem 2.4. *For the sequence $\{P_n\}_{n\geq 0}$ defined by (2.6), $\{P_{n+1} - P_n\}_{n\geq 2}$ is log-convex.*

Proof. It is obvious that

$$\begin{aligned}
R(n) &= n(n+1), \\
S(n) &= \frac{n^2(n+1)}{2}, \\
f(t) &= 2(n+2) + \frac{2(n+2)(n+3) + (n+2)^2(n+3) - 2}{2[(n+1)(n+2) - 1]t + (n+1)^2(n+2)}t \\
&\quad - \frac{2(n+1)(n+2) + (n+1)^2(n+2) - 2}{2(t-1)}, \quad (t > 1).
\end{aligned}$$

For $n \geq 2$, put $x_n = \frac{P_{n+1}}{P_n}$. Liu and Zhao [7] proved that

$$\lambda_n \leq x_n, \quad n \geq 2,$$

where $\lambda_n = n(n+1)$. It is evident that $\lambda_n > 1$ for $n \geq 2$. We note that

$$\begin{aligned}
f(\lambda_n) &= \frac{3n^3 + 6n^2 - 7n - 12}{2(n^2 + n - 1)} + \frac{[2(n+2)(n+3) + (n+2)^2(n+3) - 2]n}{2n(n^2 + 3n + 1) + (n+1)(n+2)} \\
&> 0 \quad (n \geq 2).
\end{aligned}$$

It follows from Theorem 2.1 that $\{P_{n+1} - P_n\}_{n\geq 2}$ is log-convex. □

Example 2.3. *The sequence of Motzkin numbers $\{M_n\}_{n\geq 0}$ satisfies the recurrence*

$$M_{n+1} = \frac{2n+3}{n+3}M_n + \frac{3n}{n+3}M_{n-1}, \quad n \geq 1,$$

where $M_0 = M_1 = 1$; see Table 2.3 for more values of $\{M_n\}_{n\geq 0}$. The value of M_n

n	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
M_n	*1*	*1*	*2*	*4*	*9*	*21*	*51*	*127*	*323*	*835*	*2188*

Table 2.3. *Some initial values of* $\{M_n\}_{n\geq 0}$

is the number of lattice paths from $(0,0)$ *to* (n,n)*, with steps* $(0,2)$*,* $(2,0)$*, and* $(1,1)$*, never rising above the line* $y=x$*. The sequence* $\{M_n\}_{n\geq 0}$ *is Sloane's A001006 [8]. In particular, Došlić [2] proved that* $\{M_n\}_{n\geq 0}$ *is log-convex.*

Theorem 2.5. *For the sequence of Motzkin numbers* $\{M_n\}_{n\geq 0}$*,* $\{M_{n+1}-M_n\}_{n\geq 3}$ *is log-convex.*

Proof. It is obvious that

$$R(n)=\frac{2n+3}{n+3},\quad S(n)=\frac{3n}{n+3},$$
$$f(t)=\frac{3}{(n+4)(n+5)}+\frac{4(n+2)(n+4)}{(n+1)(n+5)(t+3)}t-\frac{4(n+1)}{(n+4)(t-1)},\quad t>1.$$

For $n\geq 0$, let $x_n=\frac{M_{n+1}}{M_n}$. Došlić and Veljan [3] showed that $x_n\geq q_n$ ($n\geq 1$), where $q_n=\frac{6(n+1)}{2n+5}$. It is clear that $q_n>1$ for $n\geq 1$. Through computation, we have

$$\begin{aligned} f(q_n) &= \frac{9(4n^2-16n-47)}{(n+4)(n+5)(4n+1)(4n+7)} \\ &> 0 \quad (n\geq 6). \end{aligned}$$

It follows from Theorem 2.1 that $\{M_{n+1}-M_n\}_{n\geq 6}$ is log-convex. On the other hand, we observe that $\{M_{j+1}-M_j\}_{3\leq j\leq 7}$ is also log-convex. Hence the sequence $\{M_{n+1}-M_n\}_{n\geq 3}$ is log-convex. □

Example 2.4. *The sequence of Fine numbers* $\{f_n\}_{n\geq 0}$ *satisfies the recurrence*

$$2(n+1)f_n=(7n-5)f_{n-1}+2(2n-1)f_{n-2},\quad n\geq 2,$$

where $f_0=1$ *and* $f_1=0$*; see Table 2.4 for more values of* $\{f_n\}_{n\geq 0}$*. The value of* f_n *is the number of Dyck paths from* $(0,0)$ *to* $(2n,0)$ *with no hills. For more properties of* $\{f_n\}_{n\geq 0}$*, see Sloane's A000957 [8]. In particular, Liu and Wang [6] showed that the sequence* $\{f_n\}_{n\geq 2}$ *is log-convex.*

n	0	1	2	3	4	5	6	7	8	9	10
f_n	1	0	1	2	6	18	57	186	622	2120	7338

Table 2.4. *Some initial values of* $\{f_n\}_{n\geq 0}$

Theorem 2.6. *For the sequence of Fine numbers* $\{f_n\}_{n\geq 0}$*,* $\{f_{n+1}-f_n\}_{n\geq 3}$ *is log-convex.*

Proof. It is clear that

$$R(n)=\frac{7n+2}{2(n+2)},\quad S(n)=\frac{2(2n+1)}{2(n+2)},$$
$$f(t)=\frac{6}{(n+3)(n+4)}+\frac{9(n+2)(n+3)t}{(n+4)[(5n+3)t+2(2n+3)]}-\frac{9(n+1)}{2(n+3)(t-1)},\quad t>1.$$

For $n \geq 2$, put $x_n = \frac{f_{n+1}}{f_n}$. Liu and Wang [6] showed that $\frac{f_{n+1}}{f_n} \geq q_n$ $(n \geq 3)$, where $q_n = \frac{4n+2}{n+2}$. It is evident that $q_n > 1$ for $n \geq 3$. Through computation, we obtain

$$\begin{aligned} f(q_n) &= \frac{3(15n^3+n^2-42n-24)}{2n(n+3)(n+4)(4n^2+6n+3)} \\ &> 0 \quad (n \geq 3). \end{aligned}$$

It follows from Theorem 2.1 that the sequence $\{f_{n+1} - f_n\}_{n\geq 3}$ is log-convex. □

Example 2.5. *Let A_n denote the number of directed animals of size n. The sequence $\{A_n\}_{n\geq 0}$ is Sloane's A005773 [8] and satisfies the following recurrence*

$$A_{n+1} = 2A_n + \frac{3(n-1)}{n+1}A_{n-1}, \quad n \geq 1,$$

where $A_0 = A_1 = 1$; see Table 2.5 for more values of $\{A_n\}_{n\geq 0}$. For some properties of $\{A_n\}_{n\geq 0}$, see Exercise 6.46 of Stanley [9]. In particular, Liu and Wang [6] showed that $\{A_n\}_{n\geq 0}$ is log-convex.

n	0	1	2	3	4	5	6	7	8	9
A_n	1	1	2	5	13	35	96	267	750	2123

Table 2.5. Some initial values of $\{A_n\}_{n\geq 0}$

Theorem 2.7. *For the sequence of counting directed animals $\{A_n\}_{n\geq 0}$, $\{A_{n+1} - A_n\}_{n\geq 3}$ is log-convex.*

Proof. It is evident that

$$R(n) = 2, \quad S(n) = \frac{3(n-1)}{n+1},$$
$$f(t) = \frac{2(n+2)(2n+3)t}{(n+3)[(n+2)t+3n]} - \frac{4n+2}{(n+2)(t-1)}, \quad t > 1.$$

For $n \geq 0$, set $x_n = \frac{A_{n+1}}{A_n}$. Liu and Wang [6] proved that $x_n \geq \theta_n$ $(n \geq 0)$, where $\theta_n = \frac{6n}{2n+1}$. It is clear that $\theta_n > 1$ for $n \geq 1$. We compute that

$$\begin{aligned} f(\theta_n) &= \frac{6(2n^2-7n-13)}{(n+2)(n+3)(4n-1)(4n+5)} \\ &> 0 \quad (n \geq 5). \end{aligned}$$

It follows from Theorem 2.1 that $\{A_{n+1} - A_n\}_{n\geq 5}$ is log-convex. We observe that $\{A_{j+1} - A_j\}_{3\leq j\leq 6}$ is also log-convex. Hence the sequence $\{A_{n+1} - A_n\}_{n\geq 3}$ is log-convex. □

Example 2.6. *Let $\{a_n\}_{n\geq 1}$ denote the sequence counting directed column-convex polyominoes of height n. The sequence $\{a_n\}_{n\geq 1}$ is Sloane's A007808 [8] and satisfies the following recurrence relation*

$$a_n = (n+2)a_{n-1} - (n-1)a_{n-2}, \quad n \geq 3, \tag{2.7}$$

where $a_1 = 1$ and $a_2 = 3$; see Table 2.6 for more values of $\{a_n\}_{n\geq 1}$. In particular, Došlić [2] proved that the sequence $\{a_n\}_{n\geq 1}$ is log-convex.

n	1	2	3	4	5	6	7	8	9
a_n	1	3	13	69	431	3103	25341	231689	2345851

Table 2.6. Some initial values of $\{a_n\}_{n\geq 1}$

Theorem 2.8. *For the sequence $\{a_n\}_{n\geq 1}$ defined by (2.7), $\{a_{n+1} - a_n\}_{n\geq 1}$ is log-convex.*

Proof. For $n \geq 1$, let $x_n = \frac{a_{n+1}}{a_n}$. Došlić [2] showed that $n+2 \leq x_n \leq n+3$ for $n \geq 1$. It is clear that $n+1 > 1$ and

$$R(n) = n+3, \quad S(n) = n, \quad R(n) - 1 - S(n) > 0, \quad n \geq 1,$$

$$\Delta(n) = 1 + \frac{2}{n+3} - \frac{2}{n+1} > 0, \quad n \geq 1.$$

It follows from (i) of Theorem 2.2 that the sequence $\{a_{n+1} - a_n\}_{n\geq 1}$ is log-convex. □

Example 2.7. *Let h_n denote the number of the set of all tree-like polyhexes with $n+1$ hexagons (See Harary and Read [5]). The value of h_n is also the number of lattices paths, from $(0,0)$ to $(2n,0)$ with steps $(1,1)$, $(1,-1)$, and $(2,0)$, never falling below the x-axis and with no peaks at odd level. The sequence $\{h_n\}_{n\geq 0}$ satisfies the recurrence*

$$(n+1)h_n = 3(2n-1)h_{n-1} - 5(n-2)h_{n-2}, \quad n \geq 2,$$

where $h_0 = h_1 = 1$; see Table 2.7 for more values of $\{h_n\}_{n\geq 0}$. The sequence $\{h_n\}_{n\geq 0}$ is Sloane's A002212 [8]. In particular, Liu and Wang [6] showed that $\{h_n\}_{n\geq 0}$ is log-convex.

n	0	1	2	3	4	5	6	7	8	9
h_n	1	1	3	10	36	137	543	2219	9285	39587

Table 2.7. Some initial values of $\{h_n\}_{n\geq 0}$

Theorem 2.9. *For the sequence of counting tree-like polyhexes $\{h_n\}_{n\geq 0}$, $\{h_{n+1} - h_n\}_{n\geq 1}$ is log-convex.*

Proof. For $n \geq 0$, let $x_n = \frac{h_{n+1}}{h_n}$. Liu and Zhao [7] proved that

$$\lambda_n \leq x_n \leq \mu_n, \quad n \geq 0,$$

where $\lambda_n = \frac{10n+3}{2n+4}$ and $\mu_n = \frac{5n+4}{n+1}$. It is evident that $\lambda_n > 1$ ($n \geq 1$) and

$$R(n) = \frac{3(2n+1)}{n+2}, \quad S(n) = \frac{5(n-1)}{n+2}, \quad R(n) - 1 - S(n) > 0, \quad n \geq 1,$$

$$\Delta(n) = \frac{3(50n^2 - 54n - 257)}{(n+3)(n+4)(4n+7)(8n-1)} > 0, \quad n \geq 3.$$

It follows from (i) of Theorem 2.2 that the sequence $\{h_{n+1}-h_n\}_{n\geq 3}$ is log-convex. We find that $\{h_{j+1}-h_j\}_{1\leq j\leq 4}$ is log-convex. Thus the sequence $\{h_{n+1}-h_n\}_{n\geq 1}$ is log-convex. □

Example 2.8. *Let w_n denote the number of walks on the cubic lattice with n steps, starting and finishing on the xy plane and never going below it (See Guy [4]). The sequence $\{w_n\}_{n\geq 0}$ is Sloane's A005572 [8] and satisfies the following recurrence*

$$(n+2)w_n = 4(2n+1)w_{n-1} - 12(n-1)w_{n-2}, \quad n \geq 2, \tag{2.8}$$

where $w_0 = 1$ and $w_1 = 4$; see Table 2.8 for more values of $\{w_n\}_{n\geq 0}$. In particular, Liu and Wang [6] proved that $\{w_n\}_{n\geq 0}$ is log-convex.

n	0	1	2	3	4	5	6	7	8	9
w_n	1	4	17	76	354	1704	8421	42508	218318	1137400

Table 2.8. Some initial values of $\{w_n\}_{n\geq 0}$

Theorem 2.10. *For the sequence of counting walks on the cubic lattice $\{w_n\}_{n\geq 0}$, $\{w_{n+1}-w_n\}_{n\geq 0}$ is log-convex.*

Proof. For $n \geq 0$, let $x_n = \frac{w_{n+1}}{w_n}$. Using (2.8), we can prove by induction that

$$\lambda_n \leq x_n \leq \lambda_{n+1}, \quad n \geq 3,$$

where $\lambda_n = \frac{6n+9}{n+3}$. It is obvious that $\lambda_n > 1$ ($n \geq 3$) and

$$R(n) = \frac{4(2n+3)}{n+3}, \quad S(n) = \frac{12n}{n+3}, \quad R(n) \leq S(n) \quad (n \geq 3),$$

$$\Omega(n) = \frac{36(n+1)}{(n+4)(n+5)(5n+11)} > 0.$$

It follows from (ii) of Theorem 2.2 that the sequence $\{w_{n+1}-w_n\}_{n\geq 3}$ is log-convex. We observe that $\{w_{j+1}-w_j\}_{0\leq j\leq 4}$ is log-convex. Hence the sequence $\{w_{n+1}-w_n\}_{n\geq 0}$ is log-convex. □

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Feng-Zhen Zhao
Shanghai University
Department of Mathematics
Shanghai 200444
China
e-mail: *fengzhenzhao@shu.edu.cn*